

# Characterization of the matrix whose norm is determined by its action on decreasing sequences

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## Abstract

Let  $A = (a_{j,k})_{j,k \geq 1}$  be a non-negative matrix. In this paper, we characterize those  $A$  for which  $\|A\|_{E,F}$  are determined by their actions on decreasing sequences, where  $E$  and  $F$  are suitable normed Riesz spaces of sequences. In particular, our results can apply to the following spaces:  $\ell_p$ ,  $d(w, p)$ , and  $\ell_p(w)$ . The results established here generalize the corresponding ones given by Bennett in Quart. J. Math. Oxford (2), 49(1998), 395-432, by Chen et al in J. Math. Anal. Appl. 273(2002), 160-171 and by Jameson in Illinois J. Math. 43(1999), 79-99.

## 1 Introduction

Let  $w_1 \geq w_2 \geq \dots \geq 0$ . For  $1 \leq p \leq \infty$ , denote by  $\ell_p(w)$  the space of all sequences  $x = \{x_k\}_{k=1}^\infty$  such that

$$(1.1) \quad \|x\|_{\ell_p(w)} := \left( \sum_{k=1}^{\infty} |x_k|^p w_k \right)^{1/p} < \infty.$$

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The Lorentz sequence space  $d(w, p)$  is the space of null sequences  $x$  for which  $x^*$  is in  $\ell_p(w)$ , with norm  $\|x\|_{w,p} = \|x^*\|_{\ell_p(w)}$ , (cf. [1, 7]). Here  $x^*$  is the decreasing rearrangement of  $\{|x_k|\}_{k=1}^\infty$ . When  $w_k = 1$  for all  $k$ ,  $\ell_p(w)$  coincides with  $\ell_p$  in the usual sense (the norm of which we denote by  $\|\cdot\|_p$ ). We also have  $\ell_\infty(w) = \ell_\infty$  for any  $w$ . We write  $x \geq 0$  if  $x_k \geq 0$  for all  $k$ . Similarly,  $x \downarrow$  will mean that  $\{x_k\}_{k=1}^\infty$  is decreasing, that is,  $x_k \geq x_{k+1}$  for all  $k \geq 1$ . For a non-negative matrix  $A = (a_{j,k})_{j,k \geq 1}$  and two normed sequence spaces  $(E, \|\cdot\|_E)$ ,  $(F, \|\cdot\|_F)$  in  $\ell_p(w)$ , let  $\|A\|_{E,F}$  denote the norm of  $A$  when regarded as an operator from  $E$  to  $F$ . Clearly, for  $A \geq 0$ , the norm of  $A$  is determined by non-negative sequences and  $\|A\|_{E,F} \geq \|A\|_{E,F,\downarrow}$ , where

$$\|A\|_{E,F,\downarrow} := \sup_{\|x\|_E=1, x \geq 0, x \downarrow} \|Ax\|_F.$$

In [3, Problem 7.23], Bennett asked the following question for  $E = F = \ell_p$ : When does the equality  $\|A\|_{E,F} = \|A\|_{E,F,\downarrow}$  hold? It is one of great importance in the general theory of inequalities.

In [2, page 422] and [3, page 422], Bennett established this upper bound equality for the case that  $E = F = \ell_p$ ,  $1 < p < \infty$ , and  $A$  is a weighted mean matrix with decreasing weights  $w_n$ . This result was extended by Jameson [6, Theorem 2] to the case that  $E = F$  is a Banach lattice of sequences with property  $(PS)$  and  $A$  satisfies the following condition:

$$(1.2) \quad \sum_{j=1}^l \sum_{k=1}^r a_{j,k} \geq \sum_{j \in N_l} \sum_{k \in N_r} a_{j,k} \quad (l, r \geq 1; |N_l| = l, |N_r| = r).$$

For the definition of  $(PS)$ , we refer the readers to §3. Here  $N_s$  denotes a set of positive integers having  $s$  elements and  $|N_s| = s$  stands for all possibilities of  $N_s$ . Later, in a joint paper, the first present author extended Bennett's result in a different direction. More precisely, in [5, Lemma 2.4], Chen et al established the equality  $\|A\|_{E,F} = \|A\|_{E,F,\downarrow}$  for the case that  $E = F = \ell_p$ ,  $1 < p < \infty$ , and  $A$  is a non-negative lower triangular matrix with rows decreasing in the sense that  $a_{j,k} \geq a_{j,k+1}$  for all  $j, k \geq 1$ .

The purpose of this paper is to extend the results of Bennett, Jameson and Chen-Luor-Ou to a more general setting. In §2, we introduce the collection  $\mathcal{R}_A^{\gamma,\lambda}$ , which is a special set of row rearrangements of  $A$  with indices  $\gamma \leq \lambda$ . We prove that for a non-negative  $n \times \infty$  matrix  $A$ ,  $\mathcal{R}_A^{\gamma,\lambda} \neq \emptyset$  for some pair  $(\gamma, \lambda)$  with  $0 \leq \gamma \leq \lambda \leq n$ . We also prove that for  $x = \{x_k\}_{k=1}^\infty \geq 0$ , there exists some  $B \in \cup_{0 \leq \gamma \leq \lambda \leq n} \mathcal{R}_A^{\gamma,\lambda}$ , depending on  $A$  and  $x$ , such that the finite

sequence  $Bx = \{\sum_{k=1}^{\infty} b_{j,k}x_k\}_{j=1}^n$  is decreasing. Based on these, we establish in Theorem 3.2 the upper bound equality for the case that  $E$  and  $F$  are two suitable normed Riesz spaces of sequences with property  $(PS)$  and the following condition is satisfied by some positive integer  $n_0$ :

- (1.3) for given  $n \geq n_0$  and  $B = (b_{j,k}) \in \cup_{0 \leq \gamma \leq \lambda \leq n} \mathcal{R}_{A_n}^{\gamma, \lambda}$ , there exists some  $C \in \mathcal{R}_{A_n}$ , depending on  $n$  and  $B$ , such that the following inequality holds:

$$\sum_{j=1}^l \sum_{k=1}^r c_{j,k} \geq \sum_{j=1}^l \sum_{k \in N_r} b_{j,k} \quad (1 \leq l \leq n; r \geq 1; |N_r| = r),$$

where  $C = (c_{j,k})$ ,  $A_n$  is the  $n \times \infty$  matrix obtained from the first  $n$  rows of  $A$ , and  $\mathcal{R}_A$  is the set of all row rearrangements of  $A$ . In particular, Theorem 3.2 can apply to any of  $\ell_p$  and  $d(w, p)$  for the spaces  $E$  and  $F$ . However,  $\ell_{\infty}$  is excluded. A similar result is also established for the case  $F = \ell_p(w)$ , (cf. Theorem 3.3). In §4, we shall give a detailed investigation of (1.3) for the matrix  $A$ . These include the investigations of the Hilbert matrix, the weighted mean matrix, the Nörlund matrix, summability matrices, and matrices with row decreasing. Of course, the Gamma matrix  $\Gamma(\alpha)$  and the Cesàro matrix  $C(\alpha)$  are also examined. Since (1.2)  $\implies$  (1.3) (by choosing  $C = A_n$ ), our results generalize [6, Theorem 2] and Bennett's result. On the other hand, (1.3) is satisfied, provided  $A$  is row decreasing. In this case, we choose  $C = B$ . Therefore, our results (especially Corollary 4.7) also include [5, Lemma 2.4] as a special case. We refer the readers to §4 for details.

## 2 The collection $\mathcal{R}_A^{\gamma, \lambda}$

Let  $A = (a_{j,k})$  be an  $n \times \infty$  matrix. Here  $1 \leq j \leq n$  and  $1 \leq k < \infty$ . We say that an  $n \times \infty$  matrix  $B = (b_{j,k})$  is a matrix obtained from  $A$  by row rearrangements, if there is a one-to-one mapping  $\sigma$  from  $\{1, 2, \dots, n\}$  onto itself with  $b_{j,k} = a_{\sigma(j),k}$  for all  $j$  and for all  $k$ . Denote by  $\mathcal{R}_A$  the collection of these matrices. Clearly,  $A \in \mathcal{R}_A$ . We pay attention to the following subset of  $\mathcal{R}_A$ .

**Definition 2.1** For  $0 \leq \gamma \leq \lambda \leq n$ , we write  $B \in \mathcal{R}_A^{\gamma, \lambda}$  if and only if  $B \in \mathcal{R}_A$  and  $B = (b_{j,k})$  is of the form:

- (i)  $b_{j,k} \geq b_{j+1,k}$  for  $j \leq \gamma$  or  $j \geq \lambda$ ,
- (ii)  $b_{r_1,k} \geq b_{j,k} \geq b_{r_2,k}$  for  $r_1 \leq \gamma < j < \lambda \leq r_2$ ,

(iii) No  $\alpha$  with  $\gamma < \alpha < \lambda$  possesses the property:  $b_{\alpha,k} \geq b_{j,k}$  for all  $\gamma < j < \lambda$  or  $b_{\alpha,k} \leq b_{j,k}$  for all  $\gamma < j < \lambda$ .

By definition, no row of the matrices in  $\mathcal{R}_A^{0,\lambda}$  is greater than or equal to the other rows. Analogously, no row of the matrices in  $\mathcal{R}_A^{\gamma,n}$  is less than or equal to the other rows. Moreover, each matrix  $B = (b_{j,k})$  in  $\mathcal{R}_A^{\lambda,\lambda}$  or  $\mathcal{R}_A^{\lambda,\lambda+1}$  must be column decreasing, that is,  $b_{1,k} \geq b_{2,k} \geq \dots \geq b_{n,k}$  for all  $k$ . For  $A$  with column decreasing,  $B \in \cup_{0 \leq \gamma \leq \lambda \leq n} \mathcal{R}_A^{\gamma,\lambda}$  if and only if  $B = A$ .

**Lemma 2.2**  $\mathcal{R}_A^{\gamma,\lambda} \neq \emptyset$  for some pair  $(\gamma, \lambda)$  with  $0 \leq \gamma \leq \lambda \leq n$ .

*Proof.* We shall prove the existence of a matrix  $B = (b_{j,k})$  with  $B \in \mathcal{R}_A^{\gamma,\lambda}$  for some pair  $(\gamma, \lambda)$  obeying the condition  $0 \leq \gamma \leq \lambda \leq n$ . Fix a row  $(a_{j_1,1}, a_{j_1,2}, \dots)$  of  $A$  and check whether  $a_{j_1,k} \geq a_{j,k}$  for all  $j$  and for all  $k$  with  $j \neq j_1$ . We can consider  $j_1$  in the order:  $j_1 = 1, 2, \dots, n$ . If so, let  $(b_{1,1}, b_{1,2}, \dots) = (a_{j_1,1}, a_{j_1,2}, \dots)$  and choose another row, say  $(a_{j_2,1}, a_{j_2,2}, \dots)$ , from the other  $n-1$  rows. Check whether  $a_{j_2,k} \geq a_{j,k}$  for all  $j$  and for all  $k$  with  $j \neq j_1, j_2$ . If so, let  $(b_{2,1}, b_{2,2}, \dots) = (a_{j_2,1}, a_{j_2,2}, \dots)$  and choose another row, say  $(a_{j_3,1}, a_{j_3,2}, \dots)$ , from the other  $n-2$  rows. Check whether  $a_{j_3,k} \geq a_{j,k}$  for all  $j$  and for all  $k$  with  $j \neq j_1, j_2, j_3$ . Continue this process up to the maximal possibility. We shall stop at some step, say the  $\gamma$ th step, and we shall find the first  $\gamma$  rows of  $B$  with the property:  $b_{j,k} \geq b_{j+1,k}$  for all  $1 \leq j < \gamma$  and  $b_{\gamma,k} \geq a_{j,k}$  for all  $j \neq j_1, j_2, \dots, j_\gamma$ . Apply the same procedure to the remainder of rows in the following way. First, choose a row, say  $(a_{s_1,1}, a_{s_1,2}, \dots)$ , and check whether  $a_{s_1,k} \leq a_{j,k}$  for all  $j$  and for all  $k$  with  $j \neq j_1, j_2, \dots, j_\gamma, s_1$ . If so, let  $(b_{n,1}, b_{n,2}, \dots) = (a_{s_1,1}, a_{s_1,2}, \dots)$  and choose a new row, say,  $(a_{s_2,1}, a_{s_2,2}, \dots)$ , from the other  $n-\gamma-1$  rows of  $A$ . Check whether  $a_{s_2,k} \leq a_{j,k}$  for all  $j$  and for all  $k$  with  $j \neq j_1, \dots, j_\gamma, s_1, s_2$ . If so, let  $(b_{n-1,1}, b_{n-1,2}, \dots) = (a_{s_2,1}, a_{s_2,2}, \dots)$ . Continue this process up to the maximal possibility. We will stop at some step, which corresponds to the  $\lambda$ th row of  $B$ . We also find the last  $(n-\lambda+1)$  rows of  $B$  with the property:  $b_{j,k} \geq b_{j+1,k}$  for all  $\lambda \leq j < n$  and  $b_{\lambda,k} \leq b_{j,k}$  for all  $j \neq j_1, j_2, \dots, j_\gamma, s_1, s_2, \dots, s_{n-\lambda+1}$ . Put the rest of rows into the middle block of  $B$  in any order. Then the final matrix  $B$  has the prescribed property. This completes the proof.  $\blacksquare$

**Lemma 2.3** Let  $A = (a_{j,k})$  be a non-negative  $n \times \infty$  matrix and  $x = \{x_k\}_{k=1}^\infty \geq 0$ . Then there exists some  $B \in \cup_{0 \leq \gamma \leq \lambda \leq n} \mathcal{R}_A^{\gamma,\lambda}$ , depending on  $A$  and  $x$ , such that the sequence  $Bx = \{\sum_{k=1}^\infty b_{j,k}x_k\}_{j=1}^n$  is decreasing.

*Proof.* Lemma 2.2 guarantees the existence of a matrix  $B \in \cup_{0 \leq \gamma \leq \lambda \leq n} \mathcal{R}_A^{\gamma, \lambda}$ , say  $B \in \mathcal{R}_A^{\gamma, \lambda}$ . Let  $y_j = \sum_{k=1}^{\infty} b_{j,k} x_k$ . By Definition 2.1(i), we obtain  $y_1 \geq y_2 \geq \dots \geq y_\gamma$  and  $y_\lambda \geq y_{\lambda+1} \geq \dots \geq y_n$ . From Definition 2.1(ii), we see that  $y_\gamma \geq y_j \geq y_\lambda$  for all  $\gamma < j < \lambda$ . Make a decreasing rearrangement for  $y_{\gamma+1}, \dots, y_{\lambda-1}$ , and let  $\tilde{B}$  be the corresponding matrix by applying such row rearrangements to  $B$ . It is clear that  $\tilde{B}$  still lies in the set  $\mathcal{R}_A^{\gamma, \lambda}$  and it has the prescribed property. We complete the proof. ■

### 3 Main Results

We have the following result.

**Lemma 3.1** *Let  $\{v_k\}_{k=1}^n$  and  $\{u_k\}_{k=1}^n$  be two non-negative sequences such that*

$$(3.1) \quad \sum_{k=1}^r v_k \geq \sum_{k \in N_r} u_k \quad (r = 1, \dots, n; |N_r| = r).$$

*Then*

$$\sum_{k=1}^n v_k x_k^* \geq \sum_{k=1}^n u_k x_k \quad (x = \{x_k\}_{k=1}^n \geq 0).$$

*Proof.* We have  $x_k^* - x_{k+1}^* \geq 0$  for all  $1 \leq k < n$ . Let  $\{\tilde{u}_k\}_{k=1}^n$  denote the corresponding rearrangement of  $\{u_k\}_{k=1}^n$  such that  $\sum_{k=1}^n u_k x_k = \sum_{k=1}^n \tilde{u}_k x_k^*$ . Employing the summation by parts and (3.1), we get

$$\begin{aligned} \sum_{k=1}^n u_k x_k &= \sum_{k=1}^n \tilde{u}_k x_k^* = \sum_{k=1}^{n-1} (x_k^* - x_{k+1}^*) \left( \sum_{s=1}^k \tilde{u}_s \right) + x_n^* \left( \sum_{k=1}^n \tilde{u}_k \right) \\ &\leq \sum_{k=1}^{n-1} (x_k^* - x_{k+1}^*) \left( \sum_{s=1}^k v_s \right) + x_n^* \left( \sum_{k=1}^n v_k \right) = \sum_{k=1}^n v_k x_k^*. \end{aligned}$$

■

Let  $(F, \|\cdot\|_F)$  be a normed Riesz space of real sequences (cf. [8, p.6] for definition). Following [6], we say that  $F$  has the property  $(PS)$ , if for all  $x \in F$ ,  $x^*$  exists and the following property holds:

$$(3.2) \quad y_1^* + \dots + y_n^* \leq x_1^* + \dots + x_n^* \quad (n \geq 1) \implies y \in F \text{ and } \|y\|_F \leq \|x\|_F.$$

Clearly, for  $x \in F$ , we have  $\tilde{x} \in F$  and  $\|\tilde{x}\|_F = \|x\|_F$ , where  $\tilde{x}$  is any sequence with  $\tilde{x}^* = x^*$ . In particular,  $\tilde{x}$  can be  $x^*$  or any sequence obtained from  $x$  by

reordering  $x_k$ . We have  $x_1 + \cdots + x_n \leq x_1^* + \cdots + x_n^*$ , so the condition in (3.2) can be replaced by  $y_1^* + \cdots + y_n^* \leq x_1 + \cdots + x_n$ . Applying Lemma 3.1, we get the first main result as follows.

**Theorem 3.2** *Let  $(E, \|\cdot\|_E)$ ,  $(F, \|\cdot\|_F)$  be two normed Riesz space of real sequences with property (PS). In addition, the following property is satisfied:*

$$(3.3) \quad \|x\|_F = \lim_{n \rightarrow \infty} \|P_n x\|_F \quad (x \in F),$$

where  $P_n x$  is the projection of  $x$  onto the first  $n$  terms. Let  $A = (a_{j,k})_{j,k \geq 1}$  define an operator from  $E$  to  $F$ , given by  $Ax = y$ , where  $a_{j,k} \geq 0$  for all  $j$  and  $k$ . If (1.3) is true for some positive integer  $n_0$ , then  $\|Ax^*\|_F \geq \|Ax\|_F$  for all  $x \in E$  with  $x \geq 0$ . Hence, decreasing elements  $x$  in  $E$  are sufficient to determine  $\|A\|_{E,F}$ .

*Proof.* Let  $x \in E$  with  $x \geq 0$ . Then the (PS) property of  $E$  implies  $x^* \in E$ . Since  $A$  sends  $E$  to  $F$ , we know that  $Ax, Ax^* \in F$ . We claim that  $\|Ax^*\|_F \geq \|Ax\|_F$ . Let  $n \geq n_0$ . By Lemma 2.3, we can find  $B = (b_{j,k}) \in \mathcal{R}_{A_n}^{\gamma,\lambda}$  with  $0 \leq \gamma \leq \lambda \leq n$  such that  $\{\sum_{k=1}^{\infty} b_{jk} x_k\}_{j=1}^n$  is decreasing. Let  $C = (c_{j,k}) \in \mathcal{R}_{A_n}$  be the corresponding matrix obeying (1.3). Let  $l$  be fixed. Since  $\sum_{k=1}^r \left(\sum_{j=1}^l c_{jk}\right) \geq \sum_{k \in N_r} \left(\sum_{j=1}^l b_{jk}\right)$  for all  $r \geq 1$  and for all  $N_r$ , it follows from Lemma 3.1 that

$$\sum_{k=1}^m \left(\sum_{j=1}^l c_{jk}\right) x_k^* \geq \sum_{k=1}^m \left(\sum_{j=1}^l b_{jk}\right) x_k \quad (m \geq 1).$$

Let  $m \rightarrow \infty$  and reorder the above sums. Then we obtain

$$(3.4) \quad \sum_{j=1}^l \left(\sum_{k=1}^{\infty} c_{jk} x_k^*\right) \geq \sum_{j=1}^l \left(\sum_{k=1}^{\infty} b_{jk} x_k\right) \quad (l = 1, \dots, n).$$

For  $1 \leq j \leq n$ , set  $y_j = \sum_{k=1}^{\infty} c_{jk} x_k^*$  and  $z_j = \sum_{k=1}^{\infty} b_{jk} x_k$ . We also set  $y_j = z_j = 0$  for  $j > n$ . Since  $\{z_j\}_{j=1}^{\infty}$  is decreasing,  $z_j^* = z_j$  for all  $j$  and consequently, (3.4) can be rewritten in the form:  $\sum_{j=1}^l y_j \geq \sum_{j=1}^l z_j^*$  for all  $l \geq 1$ . On the other hand,  $P_n Ax^* \in F$  and it is of the form:  $P_n Ax^* = \{y'_1, \dots, y'_n, 0, \dots\}$ . Since  $C \in \mathcal{R}_{A_n}$ ,  $y = \{y_1, \dots, y_n, \dots\}$  can be obtained from the sequence  $\{y'_1, \dots, y'_n, 0, \dots\}$  by reordering the first  $n$  terms. The (PS) property of  $F$  implies  $y \in F$ , and therefore,  $z = \{z_1, z_2, \dots\} \in F$ . Moreover,  $\|P_n Ax^*\|_F = \|y\|_F \geq \|z\|_F$ . We have  $B \in \mathcal{R}_{A_n}$ . The same argument

as above also ensures that  $\|z\|_F = \|P_n Ax\|_F$ . Hence,  $\|P_n Ax^*\|_F \geq \|P_n Ax\|_F$ . Let  $n \rightarrow \infty$ . Then (3.3) implies  $\|Ax^*\|_F \geq \|Ax\|_F$ . Next, consider the case  $x \in E$ . Set  $\tilde{x} = \{\tilde{x}_k\}_{k=1}^\infty$ , where  $\tilde{x}_k = |x_k|$ . Then  $\tilde{x} \in E$ . Moreover,  $\tilde{x} \geq 0$  and  $\tilde{x}^* = x^* \in E$ . By the result we have proved,  $\|Ax^*\|_F = \|A\tilde{x}^*\|_F \geq \|A\tilde{x}\|_F$ . We have  $|\sum_{k=1}^\infty a_{j,k}x_k| \leq \sum_{k=1}^\infty a_{j,k}|x_k|$  for all  $j$ . Since  $F$  is a normed Riesz space,  $\|Ax\|_F \leq \|A\tilde{x}\|_F$ , and consequently,  $\|Ax^*\|_F \geq \|Ax\|_F$ . This ensures the validity of  $\|A\|_{E,F} = \|A\|_{E,F,\downarrow}$ . We complete the proof. ■

Theorem 3.2 generalizes [6, Theorem 2] and [5, Lemma 2.4]. We shall investigate them in §4.

Following the above proof, we see that Theorem 3.2 still holds for the case of complex sequences, if in addition, elements in  $E \cup F$  satisfy  $\|\tilde{x}\| = \|x\|$ , where  $\tilde{x} = \{|x_1|, |x_2|, \dots\}$  and  $\|\cdot\|$  denotes the corresponding norm in  $E$  or in  $F$ . Moreover, the assumption that  $A$  sends  $E$  to  $F$  can be removed from Theorem 3.2, whenever  $\|Ax^*\|_F$  and  $\|Ax\|_F$  make sense and satisfy

$$\|Ax^*\|_F = \lim_{n \rightarrow \infty} \|P_n Ax^*\|_F \quad \text{and} \quad \|Ax\|_F = \lim_{n \rightarrow \infty} \|P_n Ax\|_F.$$

In particular, the spaces  $E$  and  $F$  in Theorem 3.2 can be one of  $\ell_p$  ( $1 \leq p < \infty$ ) or  $d(w, p)$  ( $1 \leq p \leq \infty$ ). However, Theorem 3.2 can not apply to the case  $F = \ell_\infty$  (or  $\ell_\infty(w)$ ), in general. A counterexample is given by  $a_{2,2} = 1$ ,  $a_{j,k} = 0$  otherwise,  $x_2 = 1$ , and  $x_k = 0$  for  $k \neq 2$ . For this example,  $\|Ax^*\|_\infty < \|Ax\|_\infty$  and  $\|A\|_{\ell_p, \ell_\infty} \neq \|A\|_{\ell_p, \ell_\infty, \downarrow}$  for  $1 \leq p < \infty$ . In the following, we show that  $F$  can be  $\ell_p(w)$ , where  $1 \leq p < \infty$ . Since the case  $F = \ell_2(w)$  with  $w_n = 1/n^3$  fails the property (PS), Theorem 3.3 is not a special case of Theorem 3.2.

**Theorem 3.3** *Let  $1 \leq p < \infty$ ,  $A = (a_{j,k})_{j,k \geq 1} \geq 0$ , and  $(E, \|\cdot\|_E)$  be a normed Riesz space of real sequences with property (PS). If (1.3) is true for  $n_0 = 1$ , then  $\|Ax^*\|_{\ell_p(w)} \geq \|Ax\|_{\ell_p(w)}$  for all  $x \in E$  with  $x \geq 0$ . Hence, decreasing, non-negative elements  $x$  in  $E$  are sufficient to determine  $\|A\|_{E, \ell_p(w)}$ .*

*Proof.* We only need the following simple remark and for the same proof in theorem 3.2 apply: if  $x, y \geq 0$  and  $\sum_{j=1}^n x_j^p \leq \sum_{j=1}^n y_j^p$  for all  $n$ , then  $\|x\|_{\ell_p(w)} \leq \|y\|_{\ell_p(w)}$ . (By Abel summation)

## 4 Investigation of (1.3)

In Theorems 3.2-3.3, we point out that (1.3) is a sufficient condition for  $A$  to guarantee the validity of the equality  $\|A\|_{E,F} = \|A\|_{E,F,\downarrow}$ . The purpose of this

section is to find those conditions which are stronger than (1.3). First, we deal with conditions of Jameson type, that is, (1.2) and its related conditions. Set  $c_{j,k} = a_{j,k}$ . We see that (1.2)  $\implies$  (1.3). Here (1.3) is assumed for  $n_0 = 1$ . Moreover, the entries of  $A^t$  still satisfy (1.2), if the entries of  $A$  do. Here  $A^t$  is the transpose of  $A$ . Hence, Theorems 3.2-3.3 have the following consequence.

**Theorem 4.1** *Theorems 3.2-3.3 remain true, if (1.3) is replaced by (1.2). Moreover, the conclusions of Theorems 3.2-3.3 also hold for  $A^t$  in place of  $A$ .*

Clearly, Theorem 4.1 extends [6, Theorem 2] from  $E = F$  to any pair  $(E, F)$ . Moreover, it can apply to the case  $F = \ell_p(w)$ , (see Theorem 3.3), but [6, Theorem 2] fails to do so. We know that (4.1)  $\implies$  (1.2):

$$(4.1) \quad a_{j,k} \geq a_{j+1,k} \quad (j, k \geq 1) \quad \text{and} \quad \sum_{j=1}^l a_{j,k} \geq \sum_{j=1}^l a_{j,k+1} \quad (k, l \geq 1),$$

(see [6, Proposition 3]). Hence, Theorem 4.1 has the following consequence.

**Corollary 4.2** *Theorems 3.2-3.3 remain true, if (1.3) is replaced by (4.1). Moreover, the conclusions of Theorems 3.2-3.3 also hold for  $A^t$  in place of  $A$ .*

In [6, Proposition 3], Jameson pointed out that the following condition also implies (1.2), and so Corollary 4.2 still holds, if we replace (4.1) by (4.1\*):

$$(4.1^*) \quad a_{j,k} \geq a_{j,k+1} \quad (j, k \geq 1) \quad \text{and} \quad \sum_{k=1}^r a_{j,k} \geq \sum_{k=1}^r a_{j+1,k} \quad (j, r \geq 1).$$

We shall prove in Corollary 4.7 that the second condition in (4.1\*) is redundant.

Corollary 4.2 can apply to the Hilbert matrix  $H = (h_{j,k})_{j,k \geq 1}$ , defined by  $h_{j,k} = 1/(j+k-1)$ . It can also apply to the weighted mean matrix  $A_W^{WM} = (a_{j,k}^{WM})_{j,k \geq 1}$  and the Nörlund mean matrix  $A_W^{NM} = (a_{j,k}^{NM})_{j,k \geq 1}$ , where  $a_{j,k}^{WM} = a_{j,k}^{NM} = 0$  for  $j < k$  and

$$(4.2) \quad a_{j,k}^{WM} = w_k / (w_1 + \cdots + w_j) \quad (j \geq k),$$

$$(4.3) \quad a_{j,k}^{NM} = w_{j-k+1} / (w_1 + \cdots + w_j) \quad (j \geq k).$$

**Corollary 4.3** *Let  $w_1 > 0$  and  $w_n \geq 0$  for all  $n > 1$ . Then Theorems 3.2-3.3 remain true, if (1.3) is replaced by any of (i) and (ii):*



(i)  $A = (A_W^{WM})^t$  with  $w_n \downarrow$ .

(ii)  $A = (A_W^{NM})^t$ , where  $w_n \uparrow$  and  $w_{n+1}/w_n \leq w_n/w_{n-1}$  for all  $n$ .

Moreover, the conclusions of Theorems 3.2-3.3 also hold for  $A^t$  in place of  $A$ .

*Proof.* Obviously,  $(A_W^{WM})^t \geq 0$  and  $(A_W^{NM})^t \geq 0$ . Consider Case (i). Set  $A = (A_W^{WM})^t = (a_{j,k})_{j,k \geq 1}$ . It is easy to see that  $a_{j,k} \geq a_{j+1,k}$  for all  $j, k \geq 1$  if and only if  $w_n \downarrow$ . Moreover, we have

$$\sum_{j=1}^l a_{j,k} = \begin{cases} \frac{w_1 + \dots + w_l}{w_1 + \dots + w_k} & (l \leq k), \\ 1 & (l > k). \end{cases}$$

This implies  $\sum_{j=1}^l a_{j,k} \geq \sum_{j=1}^l a_{j,k+1}$  for all  $k, l \geq 1$ . The above argument shows that (4.1) holds. By Corollary 4.2, we get (i). Next, consider (ii). It is an consequence of the following well-known lemma. Write  $A_n = \sum_{j=1}^n a_j$  and  $B_n$  similarly. If  $(a_n/b_n)$  is increasing(or decreasing), then so is  $A_n/B_n$ . This shows that  $\sum_{j=1}^l a_{j,k+1} \leq \sum_{j=1}^l a_{j,k}$  for  $l < k$ . This inequality is also true for the case  $l \geq k$ , because  $\sum_{j=1}^l a_{j,k+1} \leq 1 = \sum_{j=1}^l a_{j,k}$ . Thus, (4.1) is satisfied. By Corollary 4.2, we get (ii). This completes the proof. ■

The conclusion of Corollary 4.3(i) for  $A^t$  and for  $E = F = \ell_p$  was established by Bennett in [2, page 422] and [3, page 422], where  $1 < p < \infty$ . For  $w_n = \binom{n+\alpha-2}{n-1}$ ,  $A_W^{WM}$  and  $A_W^{NM}$  are denoted by  $\Gamma(\alpha)$  and  $C(\alpha)$ , respectively. They are called the Gamma matrix and the Cesàro matrix, of order  $\alpha$ , (cf. [3, p.410], [4] & [9, Chapter III]). We know that  $w_n \uparrow$  for  $\alpha \geq 1$  and  $w_n \downarrow$  for  $0 < \alpha \leq 1$  (cf. [9, page 77]). Moreover, for  $\alpha \geq 1$ , we have

$$\frac{w_{n+1}}{w_n} = \frac{n + \alpha - 1}{n} \leq \frac{n + \alpha - 2}{n - 1} = \frac{w_n}{w_{n-1}}.$$

Hence, by Corollary 4.3, the conclusions of Theorems 3.2-3.3 hold for  $A$  to be any of the matrices:  $\Gamma(\alpha), \Gamma(\alpha)^t$  ( $0 < \alpha \leq 1$ ) and  $C(\alpha), C(\alpha)^t$  ( $\alpha \geq 1$ ).

Following [3], we say that  $A = (a_{j,k})_{j,k \geq 1}$  is a summability matrix, if  $A$  is a non-negative lower triangular matrix with  $\sum_{k=1}^{\infty} a_{j,k} = 1$  for all  $j$ . For such type of matrices, we have the following result.

**Corollary 4.4** *Let  $A = (a_{j,k})_{j,k \geq 1}$  be a summability matrix. Then (4.4)  $\implies$  (1.2), where*

$$(4.4) \quad a_{j,k} \geq \max(a_{j+1,k}, a_{j+1,k+1}) \quad (j \geq k \geq 1).$$

Hence, Theorems 3.2-3.3 remain true, if (1.3) is replaced by (4.4). Moreover, the conclusions of Theorems 3.2-3.3 also hold for  $A^t$  in place of  $A$ .

*Proof.* The second part follows from Theorem 4.1. We claim that (4.4)  $\implies$  (1.2). Divide the proof into three cases. Case I is  $l \leq r$ . For this case, we have

$$(4.5) \quad \sum_{j=1}^l \sum_{k=1}^r a_{j,k} \geq \sum_{j=1}^l \sum_{k=1}^l a_{j,k} = l.$$

On the other hand, we know that  $A$  is a summability matrix. Thus,  $\sum_{k=1}^{\infty} a_{j,k} = 1$  for all  $j$ . This implies

$$(4.6) \quad \sum_{j \in N_l} \sum_{k \in N_r} a_{j,k} \leq \sum_{j \in N_l} \left( \sum_{k=1}^{\infty} a_{j,k} \right) = |N_l| = l.$$

Putting (4.5) – (4.6) together yields (1.2) for Case I. Next, consider the case:  $l > r$  and  $N_r = \{1, 2, \dots, r\}$ . Write  $N_l = \{j_1, \dots, j_l\}$  in the alphabet order. Then

$$(4.7) \quad \sum_{s=1}^r \sum_{k \in N_r} a_{j_s,k} \leq \sum_{s=1}^r \left( \sum_{k=1}^{\infty} a_{j_s,k} \right) = r = \sum_{s=1}^r \sum_{k=1}^r a_{s,k}.$$

On the other hand, for  $r < s \leq l$  and  $k \in N_r$ , by (4.4), we get  $a_{s,k} \geq a_{j_s,k}$ , and so  $\sum_{k \in N_r} a_{j_s,k} \leq \sum_{k=1}^r a_{s,k}$ . Sum up both sides over  $s \in \{r+1, \dots, l\}$ . Then

$$(4.8) \quad \sum_{s=r+1}^l \sum_{k \in N_r} a_{j_s,k} \leq \sum_{s=r+1}^l \sum_{k=1}^r a_{s,k}.$$

Putting (4.7) – (4.8) together yields  $\sum_{j \in N_l} \sum_{k \in N_r} a_{j,k} \leq \sum_{s=1}^l \sum_{k=1}^r a_{s,k}$ . This is (1.2). It remains to prove the case that  $l > r$  and  $N_r$  is any set of positive integers with  $|N_r| = r$ . Write  $N_r = \{k_1, \dots, k_r\}$  in the alphabet order. We can assume that  $j_1 \geq k_1$ , otherwise,  $a_{j_1,k} = 0$  for all  $k \in N_r$ . In this case,  $\sum_{k \in N_r} a_{j_1,k} = 0$ , which allows us to replace  $a_{j_1,k}$ , with  $k \in N_r$ , by  $a_{j,k}$  for some  $j \notin N_l$ . Let  $N_l$  be the corresponding new index set. Our replacement leads us to deal with a case, which has a bigger sum on the right side of (1.2). Similarly, we can assume  $j_l \geq k_r$ . With the help of (4.4), we can replace  $a_{j_s,k_t}$  by  $a_{j_s-k_1+1,k_t-k_1+1}$ . After this replacement, we can assume  $k_1 = 1$ . We shall prove that under suitable replacements, we can assume  $k_t = t$  for all  $t = 2, \dots, r$ . For any  $t^*$  with  $k_{t^*+1} \geq k_{t^*} + 2$ , let  $s^*$  be the smallest integer with  $j_{s^*} \geq k_{t^*+1}$ . This  $s^*$  exists, because  $j_l \geq k_r \geq k_{t^*+1}$ . If  $s^* > 1$ , then  $j_{s^*-1} < k_{t^*+1}$  and so

$a_{j_s, k_t} = 0$  for all  $1 \leq s < s^*$  and  $t^* < t \leq r$ . Here we use the fact that  $A$  is a lower triangular matrix. Replace  $a_{j_s, k_t}$  by  $a_{j_s-1, k_t-1}$  for  $s^* \leq s \leq l$  and  $t^* < t \leq r$ . Simultaneously, we make the change  $a_{j_s, k_t} \longrightarrow a_{j_s-1, k_t}$  for  $s^* \leq s \leq l$  and  $1 \leq t \leq t^*$ , whenever  $j_s > j_{s-1} + 1$ . If  $s^* = 1$ , then  $j_{s^*} \geq k_{t^*+1} \geq 2$ . Replace  $a_{j_s, k_t}$  by  $a_{j_s-1, k_t-1}$  with  $s^* \leq s \leq l, t^* < t \leq r$ , and make the change  $a_{j_s, k_t} \longrightarrow a_{j_s-1, k_t}$  for all  $s^* \leq s \leq l$  and  $1 \leq t \leq t^*$ . The above argument shows that the difference  $k_{t^*+1} - k_{t^*}$  can be reduced by 1 for each replacement. Continue this process several times and we finally get  $k_{t^*+1} = k_{t^*} + 1$ . Our argument leads us to the choice  $k_t = t$  for all  $t$  and our problem reduces to Case II. This has been proved before. Hence, the desired result follows. ■

Corollary 4.4 allows us to deal with the case  $A = A_W^{NM}$  with  $w_n \downarrow$ .

**Corollary 4.5** *Let  $w_1 > 0$  and  $w_n \geq 0$  for all  $n > 1$ . Then Theorems 3.2-3.3 remain true, if (1.3) is replaced by  $A = A_W^{NM}$  with  $w_n \downarrow$ . Moreover, the conclusions of Theorems 3.2-3.3 also hold for  $(A_W^{NM})^t$  in place of  $A$ .*

*Proof.* We know that  $A_W^{NM} = (a_{j,k}^{NW})_{j,k \geq 1}$  is a summability matrix. The hypothesis that  $w_n \geq 0$  and  $w_n \downarrow$  implies

$$\frac{w_1 + \cdots + w_{j+1}}{w_1 + \cdots + w_j} \geq 1 \geq \frac{w_{j-k+2}}{w_{j-k+1}} \quad (j \geq k \geq 1).$$

This leads us to (4.4) for  $a_{j,k} = a_{j,k}^{NW}$ . By Corollary 4.4, we get the desired result. ■

For  $w_n = \binom{n+\alpha-2}{n-1}$ ,  $A_W^{NM} = C(\alpha)$ . Moreover,  $w_n \downarrow \iff 0 < \alpha \leq 1$ . Hence, by Corollary 4.5, the conclusions of Theorems 3.2-3.3 hold for  $A$  to be one of the matrices:  $C(\alpha), C(\alpha)^t$  ( $0 < \alpha \leq 1$ ).

The matrix  $A_W^{NM}$  involved in Corollary 4.5 is row increasing in the triangular sense, that is,  $a_{j,k} \leq a_{j,k+1}$  for all  $j > k$ . This fact does not imply that Corollary 4.5 can be extended to any summability matrix with rows increasing in the triangular sense. A counterexample is given below:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1/2 & 1/2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For  $p = 1$ , we have  $\|A\| = \|Ae_2\|_1 = 3/2$ , but for decreasing  $x = (x_n)$ ,

$$(4.9) \quad \begin{aligned} \|Ax\|_1 &= x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 + x_4 + \cdots \\ &\leq \frac{5}{4}(x_1 + x_2) + \cdots \leq \frac{5}{4}\|x\|_1. \end{aligned}$$

In Theorem 4.1, we deal with the condition (1.2), which corresponds to the case  $c_{j,k} = a_{j,k}$  of (1.3). In the following, we consider the case  $c_{j,k} = b_{j,k}$ . More precisely, we consider the following condition for  $n \geq n_0$ :

$$(4.10) \quad \sum_{j=1}^l \sum_{k=1}^r b_{j,k} \geq \sum_{j=1}^l \sum_{k \in N_r} b_{j,k} \quad (1 \leq l \leq n; \quad r \geq 1; \quad |N_r| = r),$$

where  $n_0$  and  $B = (b_{j,k})$  are stated in (1.3). We know that  $(4.10) \iff (4.10^*)$ :

$$(4.10^*) \quad \left\{ \sum_{j=1}^l b_{j,k} \right\}_{k=1}^{\infty} \text{ is a decreasing sequence} \quad (1 \leq l \leq n).$$

This fact can be derived by considering  $N_r = \{1, \dots, r-1, r+1\}$ . By Theorems 3.2-3.3, we obtain the following result.

**Theorem 4.6** *Theorems 3.2-3.3 remain true, if (1.3) is replaced by any of (4.10) and (4.10\*).*

The matrix  $A = (a_{j,k})_{j,k \geq 1}$ , with  $a_{1,1} = a_{2,2} = a_{2,3} = 1$  and 0 otherwise, obeys the inequality:  $\|A\|_{\ell_2, \ell_2} > \|A\|_{\ell_2, \ell_2, \downarrow}$ . This follows from the fact that

$$\|A\|_{\ell_2, \ell_2} = \sup_{\|x\|_2=1, x \geq 0} (x_1^2 + x_2^2 + x_3^2 + 2x_2x_3)^{1/2}$$

is attained only at  $x = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \dots)$ , which is not a decreasing sequence. This example shows that Theorem 4.6 is not true, if (4.10) is replaced by (4.11):

$$(4.11) \quad \sum_{j=1}^l \sum_{k=1}^r a_{j,k} \geq \sum_{j=1}^l \sum_{k \in N_r} a_{j,k} \quad (l, r \geq 1; \quad |N_r| = r).$$

Obviously, (4.11) is weaker than (1.2). We know that (4.11) is equivalent to the second part of (4.1). Hence, the first part of (4.1) can not be removed from Corollary 4.2.

For  $B = (b_{j,k}) \in \cup_{0 \leq \gamma \leq \lambda \leq n} \mathcal{R}_{A_n}^{\gamma, \lambda}$ ,  $\sum_{j=1}^l b_{j,k} = \sum_{j \in N_l} a_{j,k}$  for some index set  $N_l$  with  $|N_l| = l$ . Hence,  $(4.12) \implies (4.10^*)$ :

(4.12)  $A$  is row decreasing, that is,  $a_{j,k} \geq a_{j,k+1}$  for all  $j, k \geq 1$ .

As a consequence of Theorem 4.6, we obtain the following result.

**Corollary 4.7** *Theorems 3.2-3.3 remain true, if (1.3) is replaced by (4.12).*

Corollary 4.7 extends [5, Lemma 2.4] from the pair  $(\ell_p, \ell_p)$  to the pair  $(E, F)$ . Moreover, it indicates that the condition (5\*) in [6, Proposition 3] is enough to ensure the validity of [6, Theorem 2]. Obviously, the entries of the Hilbert matrix  $H$  satisfy (4.12). Hence, the conclusions of Theorem 3.2-3.3 hold for  $A = H$ . Applying Corollary 4.7 to the Nörlund mean matrix  $A_W^{NW}$ , we get the following consequence.

**Corollary 4.8** *Let  $w_1 > 0$  and  $w_n \geq 0$  for all  $n > 1$ . Then Theorems 3.2-3.3 remain true, if (1.3) is replaced by  $A = A_W^{NM}$  with  $w_n \uparrow$ .*

Corollary 4.8 is a generalization of Corollary 4.3(ii) for the Nörlund mean matrix  $A_W^{NM}$ . For this matrix, the condition  $w_{n+1}/w_n \leq w_n/w_{n-1}$ , required in Corollary 4.3(ii), is redundant. However, we do not know whether this condition can be removed for the transpose  $(A_W^{NM})^t$ . For the case  $w_n = \binom{n+\alpha-2}{n-1}$ , it does, (see the statement given after the proof of Corollary 4.3). It is still open for general  $w_n$ .

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